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Optimal Existence Criteria for Symmetric Positive Solutions of Third-order Singular Boundary Value Problems*

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Abstract: In this paper, based on the Leggett-Williams fixed point theorem together with constructing a special cone, some new optimal existence results of triple symmetric positive solutions of third-order singular boundary value problems are obtained. Finally, an example demonstrates the validity of the main results.

Keywords: symmetric positive solution; existence; boundary value problem

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1 Introduction

We consider the existence of symmetric positive solutions for the following third-order singular boundary value problem (BVP)

$$\begin{cases} z''' + f(t, z(t)) = 0, \quad t \in (0,1), \\ z(0) = z'(0) = z(1) = 0, \end{cases} \tag{1}$$

where $f : (0,\ 1) \times [0, +\infty) \longrightarrow [0, +\infty)$ satisfies Caratheodary conditions and $f(t, z(t))$ may be singular at $t = 0$ and/or $t = 1$.

Boundary value problems arise in a variety of different areas of applied mathematics and physics (see [1,2], and the references therein). Recently, many authors have studied the existence of positive solutions for singular boundary value problems (see [3-5] and the references therein), and there are many papers focused on third-order boundary value problems. However, successful results about the existence of symmetric positive solutions to the BVP (1) are few.

Inspired and motivated by the results in [3] and [5], this paper try to establish some optimal existence criteria for the existence of triple symmetric solutions to the BVP (1) by applying a fixed point theorem.

2 Preliminaries

We consider the BVP (1) in a Banach space $E = C[0,1]$ equipped with the norm

$$\|z\| = \max_{0 \le t \le 1} |z(t)|, \quad I = [0,1]$$

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for any $z \in E$. Let K be a cone of E, and z be a nonnegative continuous concave function on I, and m, n be constants, $0 < m < n$. Denote

$$K_r = \{z \in K \mid \|z\| < r\}, \quad \overline{K}_r = \{z \in K \mid \|z\| \leq r\},$$

$$K(u,\, m,\, n) = \{z \in K \mid m \leq u(z),\, \|z\| \leq n\}.$$

Lemma 2.1[6] Let $A : K \longrightarrow K$ be a completely continuous operator, u be a nonnegative continuous concave function on K, and satisfies $u(z) \leq \|z\|$, for all $z \in \overline{K}_r$. In addition, assume that there exist $0 < d < m < n \leq r$ satisfy the following conditions:

(i) $\{z \in K(u,\, m,\, n) \mid u(z) > m\} \neq \phi$, and $u(Az) > m$ for $z \in K(u, m, n)$;

(ii) $\|Az\| \leq d$ for $z \in \overline{K}_d$;

(iii) $u(Az) > m$ for $z \in K(u,\, m,\, r)$ and $\|Az\| > n$;

then A has at least three fixed points z_1, z_2, z_3 on $\overline{K}_r$ satisfies $\|z_1\| < d$, $m < u(z_2)$, and $\|z_3\| > d$ for $u(z_3) < m$.

Let $G(t, s)$ be the Green's function of the homogeneous linear problem of the BVP (1), that is

$$G(t,s) = \begin{cases} \frac{1}{2}s^2(1-t)^2 + s(1-t)(t-s), & 0 \leq s \leq t \leq 1, \\ \frac{1}{2}t^2(1-s)^2, & 0 \leq t \leq s \leq 1. \end{cases}$$

It is easy to prove the following properties of the Green's function:

(I) $G(\mu(s), s) = \dfrac{s(1-s)^2}{2(2-s)}$, where $G(\mu(s), s) = \max\limits_{0 \leq t \leq 1} G(t, s)$ and $\mu(s) = \frac{1}{2-s}$, $s \in [0, 1]$.

(II) $a(t)G(\mu(s), s) \leq G(t, s) \leq G(\mu(s), s)$, $(t, s) \in [0, 1] \times [0, 1]$, where $a(t) = t^2(1-t)$.

(III) $\max\limits_{0 \leq t \leq 1} \displaystyle\int_0^1 G(t, s)\mathrm{d}s = \frac{1}{3}(3\ln 2 - 2)$.

(IV) $\min\limits_{\frac{1}{4} \leq t \leq \frac{3}{4}} \displaystyle\int_{\frac{1}{4}}^{\frac{3}{4}} G(t, s)\mathrm{d}s = \int_{\frac{1}{4}}^{\frac{3}{4}} G\Big(\frac{1}{4}, s\Big)\mathrm{d}s = \frac{13}{3072}$.

(V) $\min\limits_{0 \leq c \leq 1} \dfrac{G(\frac{1}{4}, c)}{G(\frac{1}{2}, c)} = \dfrac{1}{4}$.

3 Main results

Theorem 3.1 Suppose the following conditions hold:

(H_1) $f \in C((0,1) \times [0,+\infty), [0,+\infty))$, $f(t,z) \leq g(t)h(z)$, $g \in C((0,1), [0,+\infty))$, $h \in C([0,+\infty), [0,+\infty))$;

(H_2)

$$0 < \int_0^1 G(\mu(s), s)g(s)\mathrm{d}s < +\infty;$$

(H_3) There exist $0 < d < m < \frac{r}{2}$ such that

1) $h(z) \leq d\Big[\displaystyle\int_0^1 G(\mu(s), s)g(s)\mathrm{d}s\Big]^{-1}$, for $0 \leq z \leq d$;

2) $f(t,z) > m\Big[\int_{\frac{1}{4}}^{\frac{3}{4}} G\Big(\frac{1}{4},s\Big)\mathrm{d}s\Big]^{-1} = \frac{3072}{13}m$, for $m \le z \le 2m$;

3) $h(z) < r\Big[\int_0^1 G(\mu(s),s)g(s)\mathrm{d}s\Big]^{-1}$, for $0 \le z \le r$;

then the BVP (1) has triple symmetric positive solutions z_1, z_2, z_3 satisfy $\|z_1\| < d$, $m < u(z_2)$, and $\|z_3\| > d$ for $u(z_3) < m$.

Proof Denote $C^+[0,1] = \{z \in C[0,1] \mid z(t) \ge 0,\ t \in [0,1]\}$, and

$$K = \big\{z \in C^+[0,1] \mid z \ \text{ is concave } \ z(t) = z(1-t),\ t \in [0,1]\big\}.$$

Let

$$u(z) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} z(t) \quad \text{for all} \quad z \in K.$$

Then

$$u(z) = z\Big(\frac{1}{4}\Big) \le z\Big(\frac{1}{2}\Big) = \|z\|.$$

It is well known that $z \in C[0,1] \cap C^2(0,1)$ is a positive solution of the BVP (1) if and only if $z \in C[0,1]$ is a positive solution of the equation

$$z(t) = \int_0^1 G(t,s)f(s,z(s))\mathrm{d}s.$$

Define operator $A : K \longrightarrow K$ by

$$Az(t) = \int_0^1 G(t,s)f(s,z(s))\mathrm{d}s.$$

Obviously, $Az(t) \ge 0$ for all $z \in K$, and it is easy to see $(Az)'(t) \ge 0$, for $t \in (0, \frac{1}{2}]$, $(Az)'(t) \le 0$, for $t \in [\frac{1}{2}, 1)$, and $Az(t) = Az(1-t)$ for $0 < t < 1$. Consequently $Az \in K$, that is $A : K \longrightarrow K$. By Arzela-Ascoli theorem, we can prove $A : K \longrightarrow K$ is completely continuous, see [3].

From (H_1) and 3) in (H_3), for any $z \in \overline{K}_r$, we know that

$$\begin{aligned}\|Az\| &\le \int_0^1 G(\mu(s),s)g(s)h(z(s))\mathrm{d}s \\ &\le \int_0^1 G(\mu(s),s)g(s)\mathrm{d}s \cdot r\Big[\int_0^1 G(\mu(s),s)g(s)\mathrm{d}s\Big]^{-1} = r.\end{aligned}$$

Hence $A(\overline{K}_r) \subset \overline{K}_r$.

Choose $z(t) = 2m$, $0 < t < 1$, then $z(t) \in K(u,m,2m)$, and $u(z) = u(2m) > m$. Thus $\{z \in K(u,m,2m) \mid u(z) > a\} \ne \phi$. On the other hand, for $z \in K(u,m,2m)$, we know that $u(z) = z(\frac{1}{4}) \ge m$. So $m \le z(s) \le 2m$, $\frac{1}{4} \le s \le \frac{3}{4}$. Thus for any $z \in K(u,m,2m)$, from 2) in (H_3), we get

$$\begin{aligned}u(Az) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} Az(t) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_0^1 G(t,s)f(s,z(s))\mathrm{d}s \\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} G\Big(\frac{1}{4},s\Big)f(s,z(s))\mathrm{d}s > \frac{3072}{13}m\int_{\frac{1}{4}}^{\frac{3}{4}} G\Big(\frac{1}{4},s\Big)\mathrm{d}s = m.\end{aligned}$$

Then condition (i) of Lemma 2.1 hold.

Next, condition (ii) of Lemma 2.1 follows immediately from (H_1) and 1) in (H_3). For any $z \in \overline{K}_d$, we have

$$\begin{aligned}\|Az\| &\leq \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)h(z(s))\mathrm{d}s \\ &\leq \int_0^1 G(\mu(s),s)g(s)\mathrm{d}s \cdot d\Big[\int_0^1 G(\mu(s),s)g(s)\mathrm{d}s\Big]^{-1} = d.\end{aligned}$$

Hence $A : \overline{K}_d \longrightarrow \overline{K}_d$.

Finally, we prove $u(Az) > m$ for $z \in K(u,m,r)$ and $\|Az\| > 4m$.

In fact, from 2) in (H_3), for $z \in K(u,m,r)$ and $\|Az\| > 4m$, we know that

$$\begin{aligned}u(Az) &= Az\Big(\frac{1}{4}\Big) = \int_0^1 G\Big(\frac{1}{4},s\Big)f(s,z(s))\mathrm{d}s = \int_0^1 \frac{G(\frac{1}{4},s)}{G(\frac{1}{2},s)}G\Big(\frac{1}{2},s\Big)f(s,z(s))\mathrm{d}s \\ &\geq \min_{0\leq c\leq 1}\frac{G(\frac{1}{4},c)}{G(\frac{1}{2},c)}\int_0^1 G\Big(\frac{1}{2},s\Big)f(s,z(s))\mathrm{d}s = \frac{1}{4}Az\Big(\frac{1}{2}\Big) = \frac{1}{4}\|Az\| > m.\end{aligned}$$

Therefore, the condition (iii) of Lemma 2.1 hold.

By Lemma 2.1, we conclude that A has triple symmetric fix points. Consequently, the BVP (1) has triple symmetric solutions z_1, z_2, z_3 satisfy $\|z_1\| < d$, $m < u(z_2)$ and $\|z_3\| > d$ for $u(z_3) < m$.

Remark 1 The results of Theorem 3.1 is new. Theorem 3.1 also holds when nonlinearity $f(t,z(t))$ is nonsingular at $t=0$ and $t=1$.

Theorem 3.2 Let $f(t,z(t)) = g(t)h(z(t))$ and the following conditions hold:

(H_1') $h \in C([0,+\infty),[0,+\infty))$, $g \in C((0,1),[0,+\infty))$;

(H_2') $0 < \int_0^1 G(\mu(s),s)g(s)\mathrm{d}s < +\infty$, $\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)\mathrm{d}s > 0$;

(H_3') there exist $0 < d < m < \frac{r}{2}$ such that

1) $h(z) < d\Big[\max_{t\in[0,1]}\int_0^1 G(t,s)g(s)\mathrm{d}s\Big]^{-1}$, for $0 \leq z \leq d$;

2) $h(z) > m\Big[\int_{\frac{1}{4}}^{\frac{3}{4}} G\Big(\frac{1}{4},s\Big)g(s)\mathrm{d}s\Big]^{-1}$, for $m \leq z \leq 2m$;

3) $h(z) < r\Big[\max_{t\in[0,1]}\int_0^1 G(t,s)g(s)\mathrm{d}s\Big]^{-1}$, for $0 \leq z \leq r$;

then the boundary value problem (1) has triple symmetric positive solutions z_1, z_2, z_3 satisfy $\|z_1\| < d$, $m < u(z_2)$, and $\|z_3\| > d$ for $u(z_3) < m$, where

$$u(z) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} z(t).$$

Proof The proof is similar to Theorem 3.1, we omit it here.

Corollary 3.1 Suppose that (H_1') and (H_2') hold, and 3) in (H_3') is replaced by the following condition

$$\lim_{z\to+\infty}\frac{h(z)}{z}<\Big[\max_{t\in[0,1]}\int_0^1 G(t,s)g(s)\mathrm{d}s\Big]^{-1}.$$

Then the boundary value problem (1) has triple symmetric positive solutions.

4 Examples

Example 4.1 The following boundary value problem

$$\begin{cases} z'''+\frac{h(z)}{t(1-t)}=0, & t\in[0,1],\\ z(0)=z(1)=z'(0)=0, \end{cases} \tag{2}$$

has triple symmetric positive solutions, where

$$h(z)=\begin{cases} 4z^2, & 0\le z<2,\\ \frac{763}{2z-1}, & z\ge 2. \end{cases}$$

Proof Let

$$g(t)=\frac{1}{t(1-t)}.$$

Obviously $g(t)$ is singular at $t=0$ and $t=1$. $h(z)$ is continuous on $[0,+\infty)$. So condition (H_1') holds.

Since

$$\int_0^1 G(\mu(s),s)g(s)\mathrm{d}s=\frac{1}{2}(1-\ln 2)<+\infty,\quad \int_{\frac14}^{\frac34} g(s)\mathrm{d}s<+\infty,$$

then (H_2') holds.

1) in (H_3') follows immediately from

$$\Big(\max_{t\in[0,1]}\int_0^1 G(t,s)g(s)\mathrm{d}s\Big)^{-1}=\Big(\int_0^1 G(\mu(s),s)g(s)\mathrm{d}s\Big)^{-1}=\frac{2}{1-\ln 2},$$

$$\Big(\int_{\frac14}^{\frac34} G\Big(\frac14,s\Big)g(s)\mathrm{d}s\Big)^{-1}=\Big(\int_{\frac14}^{\frac34}\frac{1}{32}(1-s)^2\frac{1}{s(1-s)}\mathrm{d}s\Big)^{-1}=\frac{32}{\ln 3-0.5},$$

we may take $d=\frac14$, then

$$h(z)=4z^2\le 4d^2=\frac14<\frac{2}{4(1-\ln 2)}\quad\text{for}\quad 0\le z\le d=\frac14.$$

2) in (H_3') is immediate, since we may take $m=2$, then

$$h(z)=\frac{763}{2z-1}\ge h(4)=109>2\times\frac{32}{\ln 3-\frac12},\quad 2\le z\le 4.$$

3) in (H_3') is immediate since we may take $r=100>2m=4$, then

$$\max_{z\in[0,100]}h(z)\le h(2)=\frac{763}{3}<100\times\frac{2}{1-\ln 2},\quad 0\le z\le d=100.$$

Thus, from Theorem 3.2, we know that the BVP (2) has triple symmetric positive solution z_1, z_2, z_3 satisfy $\|z_1\| < \frac{1}{4}$, $2 < u(z_2)$, and $\|z_3\| > \frac{1}{4}$ for $u(z_3) < 2$.

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三阶奇异边值问题对称正解的最优存在性

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摘 要: 本文利用Leggett-Williams不动点定理，通过构造特殊的锥，得到了三阶奇异边值问题三个对称正解最优存在性的新结果，最后，通过具体的例子说明了我们所得结果的有效性。

关键词: 对称正解；存在性；边值问题